Constructive Topology and The Basic Perturbation Lemma in HoTT

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1 Traditional Preliminaries

Before launching into the definitions specific to Effective Algebraic Topology (EAT) and Homological Perturbation Theory, we need to take an unexpectedly deep dive into the true meaning of homotopy and properties of double complexes and the category of chain complexes. This is because when it comes to describing the fundamental object of Homological Perturbation Theory, the *reduction* (also known as *Strong Deformation Retract Data*, or SDR-data), the expanded-out definition presented in the paper Cat.pdf does not fully explain how this concept relates to the usual notion of a strong deformation retract.

Note: The majority of this section's beginning was composed after reading many articles on nLab and filling in the computational/proof details that were left out by their slick, high-level presentation. It's assumed that the reader is already familiar with basic homology and homotopy theory. Later in this section, where we talk about Strong Deformation Retracts, some of the additional conditions we can impose on the data behind such retracts are drawn from the paper Constructive Algebraic Topology by Julio Rubio and Francis Sergeraert. The same is true for the processes for taking a general SDR and forcing it to satisfy these additional conditions. However, based on my reading, the *interpretations* of each of these conditions are not included in the original paper.

1.1 Categories of Chain Complexes

Fix some ring R, and consider some subcategory A of the category of R-modules ($R-Mod$) which contains a zero object 0. For our purposes, a *chain complex* over A is an infinite collection of objects of A, $\{X_n\}_{n\in\mathbb{N}}$ together with a collection of morphisms (called *differentials* or *boundary operators*) $\{\partial_n^X : X_{n+1} \to X_n\}_{n\in\mathbb{N}}$ in A such that for any $n \in \mathbb{N}$, $\partial_n^X \partial_{n+1}^X = 0$. Visually, they look like this:

$$\ldots \longrightarrow X_{n+1} \xrightarrow{\partial_n^X} X_n \xrightarrow{\partial_{n-1}^X} \ldots \longrightarrow X_1 \xrightarrow{\partial_0^X} X_0 \longrightarrow \mathbf{0}$$

Given two chain complexes X and Y, a *chain map* between them is a collection of R-module homomorphisms $f_n : X_n \to Y_n$ such that every square in the diagram:

$$\begin{tikzcd}
\ldots \arrow[r] & X_{n+1} \arrow[r, "\partial_n^X"] \arrow[d, "f_{n+1}"] & X_n \arrow[r, "\partial_{n-1}^X"] \arrow[d, "f_n"] & \ldots \arrow[r] & X_1 \arrow[r, "\partial_0^X"] \arrow[d, "f_1"] & X_0 \arrow[r] \arrow[d, "f_0"] & \mathbf{0} \\
\ldots \arrow[r] & Y_{n+1} \arrow[r, "\partial_n^Y"] & Y_n \arrow[r, "\partial_{n-1}^Y"] & \ldots \arrow[r] & Y_1 \arrow[r, "\partial_0^Y"] & Y_0 \arrow[r] & \mathbf{0}
\end{tikzcd}$$

commutes.

Now, given such an A, it's easy to see that the collection of all chain complexes over A together with chain maps between them forms a category $Ch(A)$. It's important to note that the category $Ch(A)$ actually is a sub-category of the category of R-modules, since every chain complex X comes with its own module structure by viewing it as the direct sum $\bigoplus_{n\in\mathbb{N}} X_n$ and since chain maps are linear on each X_n, they're linear on the whole complex.

We'll say that in the case where $A = R - Mod$ that we're talking about the *category of chain complexes over the ring* R, denoted (somewhat abusively) $Ch(R)$.

One important object in any category of chain complexes over the ring R is the *interval object* I, which is given by the standard simplicial chain complex on the interval:

$$\ldots \longrightarrow \mathbf{0} \longrightarrow R \xrightarrow{a\mapsto(a,-a)} R \oplus R \longrightarrow \mathbf{0}$$

1.2 The Suspension of a Chain Complex

In classical homotopy theory, there's a suspension functor which takes a topological space and transforms it by taking a cylinder over the original space collapsing the top/bottom "lids" to points. However, this construction is very geometrical in nature, and so it's difficult to adapt to the purely-algebraic setting of chain complexes. Equivalently, the classical suspension may be understood as a space comprised of a north and south pole such that there's a natural assignment of paths from every point in the original space to these poles. The second definition essentially describes the suspension as a *homotopy pushout*, but since we don't yet have a notion of homotopy available in categories of chain complexes, we'll push discussion of how the soon-to-be-introduced construction relates to this more general definition into an appendix. So instead, we'll do something seemingly arbitrary which only vaguely looks like the second approach, and then demonstrate later that it is, in fact, a suspension object in its usual categorical sense.

If $X : Ch(A)$, we can define its *suspension* ΣX by:

$$(\Sigma X)_{n+1} = X_n$$

$$(\Sigma X)_0 = \mathbf{0}$$

$$\partial_{n+1}^{\Sigma X} = -\partial_n^X$$

$$\partial_0^{\Sigma X} = \mathbf{0}$$

Intuitively, taking the suspension of a chain complex raises the dimension of all of its graded components and flips the sign of the differential. (The origin of this sign change in the differential is pushed to the appendix.) A consequence of this is that on homology, $H_{n+1}(X) = H_n(\Sigma X)$.

As a result of this definition, there's a collection of maps $s_n^X : X_n \to (\Sigma X)_{n+1}$ which are each actually the identity map (after expanding definitions), but when these are collected together (with a direct sum) into a morphism between X and ΣX in the category of chain complexes, the result is called the *shift operator* s^X. It's important to note that while the shift operator is linear, it is *not* a chain map.

1.3 Double Complexes and Bicomplexes

Note: nLab and the resource Cat.pdf conflict on their choice of terminology for these two distinct concepts. nLab takes the two terms to be synonymous, with both requiring commuting squares, but Cat.pdf adopts the convention that "bicomplex" implies anti-commuting squares. We adopt Cat.pdf's convention to be able to distinguish the two concepts.

Given the note above that $Ch(A)$ is a subcategory of $R-Mod$, we can consider the category $Ch(Ch(A))$, or the category of *double complexes* over A. As a first step at expansion, the definition of a double-complex says that we have a sequence of chain complexes $\{C_n\}_{n\in\mathbb{N}}$ and a collection of chain maps $\{\partial_n^C : C_{n+1} \to C_n\}_{n\in\mathbb{N}}$ which square to the zero chain-map. Expanding the definition of a chain complex a second time, we obtain the following definition:

A double complex is a collection $\{X_{i,j}\}_{i,j\in\mathbb{N}}$ of R-Modules such that for all $i,j \in \mathbb{N}$, there are maps $\partial_{i,j}^V : X_{i+1,j} \to X_{i,j}$ and $\partial_{i,j}^H : X_{i,j+1} \to X_{i,j}$ such that $\partial^H \circ \partial^H = 0$, $\partial^V \circ \partial^V = 0$ and $\partial^V \circ \partial^H = \partial^H \circ \partial^V$ (true at all indices, which are omitted for clarity). Equivalently, a double complex is one big commuting diagram:

$$
\begin{tikzcd}
 & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\
\cdots \arrow[r] & X_{m+1,n+1} \arrow[r, "\partial^H"] \arrow[d, "\partial^V"] & X_{m+1,n} \arrow[r, "\partial^H"] \arrow[d, "\partial^V"] & \cdots \arrow[r] \arrow[d] & X_{m+1,1} \arrow[r, "\partial^H"] \arrow[d, "\partial^V"] & X_{m+1,0} \arrow[r] \arrow[d, "\partial^V"] & \mathbf{0} \\
\cdots \arrow[r] & X_{m,n+1} \arrow[r, "\partial^H"] \arrow[d, "\partial^V"] & X_{m,n} \arrow[r, "\partial^H"] \arrow[d, "\partial^V"] & \cdots \arrow[r] \arrow[d] & X_{m,1} \arrow[r, "\partial^H"] \arrow[d, "\partial^V"] & X_{m,0} \arrow[r] \arrow[d, "\partial^V"] & \mathbf{0} \\
\cdots \arrow[r] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \mathbf{0} \\
\cdots \arrow[r] & X_{1,n+1} \arrow[r, "\partial^H"] \arrow[d, "\partial^V"] & X_{1,n} \arrow[r, "\partial^H"] \arrow[d, "\partial^V"] & \cdots \arrow[r] \arrow[d] & X_{1,1} \arrow[r, "\partial^H"] \arrow[d, "\partial^V"] & X_{1,0} \arrow[r] \arrow[d, "\partial^V"] & \mathbf{0} \\
\cdots \arrow[r] & X_{0,n+1} \arrow[r, "\partial^H"] \arrow[d] & X_{0,n} \arrow[r, "\partial^H"] \arrow[d] & \cdots \arrow[r] \arrow[d] & X_{0,1} \arrow[r, "\partial^H"] \arrow[d] & X_{0,0} \arrow[r] \arrow[d] & \mathbf{0} \\
 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} &
\end{tikzcd}
$$

such that any two steps to the right or any two steps downward yields the zero map.

Now, suppose that we wanted to take a double complex and construct a chain complex over A which somehow combines the information we get from the horizontal and vertical differentials into a single nice package. Intuitively, we could consider doing this by packaging together all of the R-modules along the diagonals pictured here with a direct sum:

$$
\begin{tikzcd}
 & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\
\cdots \arrow[r] & X_{n+1,n+1} \arrow[r, "\partial^H"] \arrow[d, "\partial^V"] & X_{n+1,n} \arrow[r, "\partial^H"] \arrow[d, "\partial^V"] & \cdots \arrow[r] \arrow[d] & X_{n+1,1} \arrow[r, "\partial^H"] \arrow[d, "\partial^V"] & X_{n+1,0} \arrow[r] \arrow[d, "\partial^V"] & \mathbf{0} \\
\cdots \arrow[r] & X_{n,n+1} \arrow[r, "\partial^H"] \arrow[d, "\partial^V"] & X_{n,n} \arrow[r, "\partial^H"] \arrow[d, "\partial^V"] & \cdots \arrow[r] \arrow[d] & X_{n,1} \arrow[r, "\partial^H"] \arrow[d, "\partial^V"] & X_{n,0} \arrow[r] \arrow[d, "\partial^V"] & \mathbf{0} \\
\cdots \arrow[r] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \mathbf{0} \\
\cdots \arrow[r] & X_{1,n+1} \arrow[r, "\partial^H"] \arrow[d, "\partial^V"] & X_{1,n} \arrow[r, "\partial^H"] \arrow[d, "\partial^V"] & \cdots \arrow[r] \arrow[d] & X_{1,1} \arrow[r, "\partial^H"] \arrow[d, "\partial^V"] & X_{1,0} \arrow[r] \arrow[d, "\partial^V"] & \mathbf{0} \\
\cdots \arrow[r] & X_{0,n+1} \arrow[r, "\partial^H"] \arrow[d] & X_{0,n} \arrow[r, "\partial^H"] \arrow[d] & \cdots \arrow[r] \arrow[d] & X_{0,1} \arrow[r, "\partial^H"] \arrow[d] & X_{0,0} \arrow[r] \arrow[d] & \mathbf{0} \\
 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} &
\end{tikzcd}
$$

and then taking the differential to be the linear map which takes elements a in the copy of $X_{m+1,n+1}$ to both $\partial^H a$ in the copy of $X_{m+1,n}$ and $\partial^V a$ in the copy of $X_{m,n+1}$. However, this does not work, because this "differential" doesn't square to zero! In particular, applying this "differential" twice to such an a in the copy of $X_{m+1,n+1}$ would yield zero everywhere except for the term $\partial^H \partial^V a + \partial^V \partial^H a$ in $X_{m,n}$.

As a result, we have a good motivation for considering a structure where $\partial^H \partial^V a = -\partial^V \partial^H a$ everywhere in the diagram. This structure, which is like a double-complex, but where every square anti-commutes instead of commuting, is called a *bicomplex*.

In a bicomplex X, the above construction of a new chain complex in $Ch(A)$ just *works*, because we *forced* the differential to square to zero. This new chain complex is called the *total complex of the bicomplex* X, and is denoted $T(X)$.

At first, bicomplexes may not seem very natural, and the connection to double-complexes may not be very clear. To fix this, we can use something called the *suspension trick* to transform any double complex into a bicomplex. The key realization is that in the total complex, we'll want all of the elements along diagonal lines to have the same degree. So we'll just progressively "bump up" the dimension of chain complexes from right-to-left using repeated suspensions to obtain the following, where every row indicates a collection of direct summands (which correspond with the diagonal lines in the diagram above) and the transitions from rows to rows will correspond to the boundary maps of the total complex by taking the direct sum of all maps between the rows.

$$
\begin{tikzcd}
\cdots \arrow[dr] & \cdots \arrow[d] & & & & & \\
\cdots \arrow[dr] & {[\Sigma^{n+1}X_{n+1}]_{m+1}} \arrow[d, "\partial^{\Sigma^{n+1}}"] \arrow[dr, "\partial^H"] & \cdots \arrow[d] & & & & \\
\cdots \arrow[dr] & {[\Sigma^{n+1}X_{n+1}]_{m}} \arrow[d, "\partial^{\Sigma^{n+1}}"] \arrow[dr, "\partial^H"] & {[\Sigma^{n}X_{n}]_{m+1}} \arrow[d, "\partial^{\Sigma^{n}}"] \arrow[dr, "\partial^H"] & \cdots \arrow[d] & & & \\
\cdots \arrow[dr] & \cdots \arrow[d] \arrow[dr] & {[\Sigma^{n}X_{n}]_{m}} \arrow[d, "\partial^{\Sigma^{n}}"] \arrow[dr, "\partial^H"] & \cdots \arrow[d] \arrow[dr] & \cdots \arrow[d] & & \\
\cdots \arrow[dr] & {[\Sigma^{n+1}X_{n+1}]_{1}} \arrow[d, "\partial^{\Sigma^{n+1}}"] \arrow[dr, "\partial^H"] & \cdots \arrow[d] \arrow[dr] & \cdots \arrow[d] \arrow[dr] & {[\Sigma X_{1}]_{n+1}} \arrow[d, "\partial^{\Sigma^{1}}"] \arrow[dr, "\partial^H"] & \cdots \arrow[d] & \\
(\text{Dim } n+1) & {[\Sigma^{n+1}X_{n+1}]_{0}} \arrow[d] \arrow[dr, "\partial^H"] & {[\Sigma^{n}X_{n}]_{1}} \arrow[d, "\partial^{\Sigma^{n}}"] \arrow[dr, "\partial^H"] & \cdots \arrow[d] \arrow[dr] & {[\Sigma X_{1}]_{n}} \arrow[d, "\partial^{\Sigma^{1}}"] \arrow[dr, "\partial^H"] & X_{n+1,0} \arrow[d, "\partial^V"] \arrow[dr] & \\
(\text{Dim } n) \arrow[d, "\cdots"] & \mathbf{0} \arrow[d, "\cdots"] & {[\Sigma^{n}X_{n}]_{0}} \arrow[d, "\cdots"] \arrow[dr, "\cdots"] & \cdots \arrow[d, "\cdots"] \arrow[dr, "\cdots"] & \cdots \arrow[d, "\cdots"] \arrow[dr, "\cdots"] & X_{n,0} \arrow[d, "\cdots"] \arrow[dr, "\cdots"] & \mathbf{0} \\
(\text{Dim } 2) & \mathbf{0} \arrow[d] & \mathbf{0} \arrow[d] & \cdots \arrow[d] \arrow[dr] & {[\Sigma X_{1}]_{1}} \arrow[d, "\partial^{\Sigma^{1}}"] \arrow[dr, "\partial^H"] & \cdots \arrow[d] \arrow[dr] & \mathbf{0} \\
(\text{Dim } 1) & \mathbf{0} \arrow[d] & \mathbf{0} \arrow[d] & \mathbf{0} \arrow[d] & {[\Sigma X_{1}]_{0}} \arrow[d] \arrow[dr, "\partial^H"] & X_{1,0} \arrow[d, "\partial^V"] \arrow[dr] & \mathbf{0} \\
(\text{Dim } 0) & \mathbf{0} \arrow[d] & \mathbf{0} \arrow[d] & \mathbf{0} \arrow[d] & \mathbf{0} \arrow[d] & X_{0,0} \arrow[d] \arrow[dr] & \mathbf{0} \\
 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}
\end{tikzcd}
$$

Where here, X_i denotes the chain complex in the ith column of the double complex X (so, e.g. $[\Sigma^{n+1}X_{n+1}]_n$ actually was derived from $X_{n,n+1}$ in the original complex). Since the differential on the suspension of a complex just flips the sign on the differential on the original (and adjusts the dimension appropriately), we essentially have

$$\partial^{\Sigma^n} = (-1)^n \partial^V$$

so the sign of the vertical differentials flips on every other column when compared with the original double complex.

If we take the data from the diagonal lattice above the triangle of $\mathbf{0}$s, and shear it back to a square shape, we get an induced bicomplex:

$$\begin{tikzcd}
 & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\
\cdots \arrow[r] & X_{m+1,n+1} \arrow[r, "\partial^H"] \arrow[d, "(-1)^{n+1}\partial^V"] & X_{m+1,n} \arrow[r, "\partial^H"] \arrow[d, "(-1)^{n}\partial^V"] & \cdots \arrow[r] \arrow[d] & X_{m+1,1} \arrow[r, "\partial^H"] \arrow[d, "-\partial^V"] & X_{m+1,0} \arrow[r] \arrow[d, "\partial^V"] & \mathbf{0} \\
\cdots \arrow[r] & X_{m,n+1} \arrow[r, "\partial^H"] \arrow[d, "(-1)^{n+1}\partial^V"] & X_{m,n} \arrow[r, "\partial^H"] \arrow[d, "(-1)^{n}\partial^V"] & \cdots \arrow[r] \arrow[d] & X_{m,1} \arrow[r, "\partial^H"] \arrow[d, "-\partial^V"] & X_{m,0} \arrow[r] \arrow[d, "\partial^V"] & \mathbf{0} \\
\cdots \arrow[r] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \mathbf{0} \\
\cdots \arrow[r] & X_{1,n+1} \arrow[r, "\partial^H"] \arrow[d, "(-1)^{n+1}\partial^V"] & X_{1,n} \arrow[r, "\partial^H"] \arrow[d, "(-1)^{n}\partial^V"] & \cdots \arrow[r] \arrow[d] & X_{1,1} \arrow[r, "\partial^H"] \arrow[d, "-\partial^V"] & X_{1,0} \arrow[r] \arrow[d, "\partial^V"] & \mathbf{0} \\
\cdots \arrow[r] & X_{0,n+1} \arrow[r, "\partial^H"] \arrow[d] & X_{0,n} \arrow[r, "\partial^H"] \arrow[d] & \cdots \arrow[r] \arrow[d] & X_{0,1} \arrow[r, "\partial^H"] \arrow[d] & X_{0,0} \arrow[r] \arrow[d] & \mathbf{0} \\
 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} &
\end{tikzcd}$$

From which we get the same total complex as the one described visually by the parallelogram lattice above. Expressed simply, the total complex of the double complex X is given by:

$$T(X)_n = \bigoplus_{i+j=n} X_{ij}$$

$$\partial_n^{T(X)}|_{X_{ij}} = (-1)^j \partial^V + \partial^H \, , \; i+j=n$$

1.4 Tensor Product of Chain Complexes

Given two chain complexes X and Y in $Ch(R)$, we could forget all of their structure as chain complexes and consider the tensor product of R-Modules

$$(\bigoplus_n X_n) \otimes (\bigoplus_m Y_m) \simeq \bigoplus_{n,m} X_n \otimes Y_m$$

but it's easy to see that we can use $1_X \otimes \partial^Y$ and $\partial X \otimes 1_Y$ to form the double complex:

$$
\begin{tikzcd}
 & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \cdots \arrow[d] & \\
\cdots \arrow[r] & X_{n+1} \otimes Y_{m+1} \arrow[r, "\partial^X \otimes 1"] \arrow[d, "1 \otimes \partial^Y"] & X_n \otimes Y_{m+1} \arrow[r, "\partial^X \otimes 1"] \arrow[d, "1 \otimes \partial^Y"] & \cdots \arrow[r] \arrow[d] & X_1 \otimes Y_{m+1} \arrow[r, "\partial^X \otimes 1"] \arrow[d, "1 \otimes \partial^Y"] & X_0 \otimes Y_{m+1} \arrow[r] \arrow[d, "1 \otimes \partial^Y"] & \mathbf{0} \\
\cdots \arrow[r] & X_{n+1} \otimes Y_m \arrow[r, "\partial^X \otimes 1"] \arrow[d, "1 \otimes \partial^Y"] & X_n \otimes Y_m \arrow[r, "\partial^X \otimes 1"] \arrow[d, "1 \otimes \partial^Y"] & \cdots \arrow[r] \arrow[d] & X_1 \otimes Y_m \arrow[r, "\partial^X \otimes 1"] \arrow[d, "1 \otimes \partial^Y"] & X_0 \otimes Y_m \arrow[r] \arrow[d, "1 \otimes \partial^Y"] & \mathbf{0} \\
\cdots \arrow[r] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \cdots \arrow[r] \arrow[d] & \mathbf{0} \\
\cdots \arrow[r] & X_{n+1} \otimes Y_1 \arrow[r, "\partial^X \otimes 1"] \arrow[d, "1 \otimes \partial^Y"] & X_n \otimes Y_1 \arrow[r, "\partial^X \otimes 1"] \arrow[d, "1 \otimes \partial^Y"] & \cdots \arrow[r] \arrow[d] & X_1 \otimes Y_1 \arrow[r, "\partial^X \otimes 1"] \arrow[d, "1 \otimes \partial^Y"] & X_0 \otimes Y_1 \arrow[r] \arrow[d, "1 \otimes \partial^Y"] & \mathbf{0} \\
\cdots \arrow[r] & X_{n+1} \otimes Y_0 \arrow[r, "\partial^X \otimes 1"] \arrow[d] & X_n \otimes Y_0 \arrow[r, "\partial^X \otimes 1"] \arrow[d] & \cdots \arrow[r] \arrow[d] & X_1 \otimes Y_0 \arrow[r, "\partial^X \otimes 1"] \arrow[d] & X_0 \otimes Y_0 \arrow[r] \arrow[d] & \mathbf{0} \\
 & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} &
\end{tikzcd}
$$

Taking the total complex of this double complex yields the *tensor product* $X \otimes Y$ of X and Y, which may be explicitly described by:

$$(X \otimes Y)_n = \bigoplus_{i+j=n} X_i \otimes Y_j$$

$$\partial_n^{X \otimes Y}|_{X_i \otimes Y_j} = (-1)^i 1 \otimes \partial^Y + \partial^X \otimes 1 \, , \; i + j = n$$

It's an important result in algebraic topology that if $C_*(X)$ is used to denote the simplicial chain complex of the simplicial space X, the homology groups of $C_*(X) \otimes C_*(Y)$ agree with the homology groups of $C_*(X \times Y)$. (This was shown by Eilenberg, Zilber and Maclane. The details of the proof are not important for our purposes.)

1.5 Chain Homotopies

We're now ready to talk about homotopy theory in the category of chain complexes. In any Algebraic Topology class, the concept of a chain homotopy will eventually pop up. Much of the time, these are defined as follows:

If you have two chain maps $f, g : C \to D$ between the chain complexes C and D, then a *chain homotopy* from f to g is a sequence of abelian group homomorphisms $h_n : C_n \to D_{n+1}$ such that $f - g = \partial h + h\partial$. However, while this is a simple definition, it does not appear to be immediately related to "homotopy" at all!

Recall that a homotopy between two morphisms $f, g : A \to B$ in Top is any map η fitting into the diagram:

$$
\begin{tikzcd}
A \arrow[d, "d_{0A}"'] \arrow[dr, "f"] & \\
I \times A \arrow[r, "\eta"] & B \\
A \arrow[u, "d_{1A}"] \arrow[ur, "g"'] &
\end{tikzcd}
$$

Where d_{0A}, d_{1A} are the maps $a \mapsto (0, a)$ and $a \mapsto (1, a)$, respectively.

1.5.1 Cylinder Functors

To take the first step toward porting this definition to a category of chain complexes, we need a suitably abstract replacement for the cartesian product of a space with an interval. $I \times A$ is usually known as the *cylinder over* A, which we'll denote $Cyl(A)$. Then the diagram above becomes:

$$
\begin{tikzcd}
A \arrow[d, "d_{0A}"'] \arrow[dr, "f"] & \\
Cyl(A) \arrow[r, "\eta"] & B \\
A \arrow[u, "d_{1A}"] \arrow[ur, "g"'] &
\end{tikzcd}
$$

Note also that if we have an arrow $f : A \to B$ in Top, we can construct a new function $Cyl(f) : Cyl(A) \to Cyl(B)$ by leaving the interval coordinate unchanged and applying f to the other coordinate. $Cyl(-)$ also trivially respects composition of functions and identities, and so it induces the *cylinder functor* $Cyl : Top \to Top$.

A functor without laws isn't very useful, so we should attempt to impose additional structure on Cyl. First, note that in the simple case of a cylinder in Top, we could define a projection $\sigma_A : I \times A \to A$ onto the second coordinate. Using this, it's easy to see that it's irrelevant whether we project first and then apply f, or apply $Cyl(f)$ (which is f on the second coordinate) and then project down. As a result, the diagram:

$$
\begin{tikzcd}
Cyl(A) \arrow[r, "Cyl(f)"] \arrow[d, "\sigma_A"'] & Cyl(B) \arrow[d, "\sigma_B"] \\
A \arrow[r, "f"] & B
\end{tikzcd}
$$

commutes, and so σ is a natural transformation from Cyl to Id. We can also notice a similar result for the $d_{0A}, d_{1A} : A \to Cyl(A)$ over by noting that for each d_i

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "d_{iA}"'] & B \arrow[d, "d_{iB}"] \\
A \arrow[r, "Cyl(f)"] & B
\end{tikzcd}
$$

commutes. Furthermore, since projection destroys the information about the first coordinate, we also have the identities $\sigma d_0 = \sigma d_1 = Id$.

Using this, we can say that a category C has a cylinder functor if there's an endofunctor Cyl such that we also have natural transformations $\sigma : Cyl \to Id$, $d_i : Id \to Cyl, i \in \{0, 1\}$ such that they satisfy the condition $\sigma d_0 = \sigma d_1 = Id$, or equivalently, such that the following diagram commutes:

$$\begin{tikzcd}
A \arrow[d, "d_{0A}"'] \arrow[dr, "Id_A"] & \\
Cyl(A) \arrow[r, "\sigma_A"] & A \\
A \arrow[u, "d_{1A}"] \arrow[ur, "Id_A"'] &
\end{tikzcd}$$

From this diagram, it's apparent that σ_A always gives a homotopy from Id_A to Id_A.

1.5.2 A Cylinder Functor in the Category of Chain Complexes

With enough cognitive impairment or intense squinting, we could hope that the definition of the cylinder object $I \times A$ in Top could be replaced by $I \otimes A$ in $Cpx(R)$. This turns out to work, but it takes some work and elaboration.

First, since all of the morphisms in $Cpx(R)$ are chain maps, it will be important to concretely define the action of the boundary map in $I \otimes A$ to verify that all maps constructed from here on out really are chain maps. First, note that

$$(I \otimes A)_k \equiv \bigoplus_{i+j=k} I_i \otimes A_j = R^2 \otimes A_k \oplus R \otimes A_{k-1}$$

The action of the boundary operator $\partial^{I \otimes A}$ adds for every $i \otimes a$ in each direct summand of $(I \otimes A)_k$ both $\partial^I i \otimes a$ and $(-1)^{deg(i)} i \otimes \partial^A a$ to the components of appropriate grades, so we should expect exactly one cross-term from a copy of $R \otimes A_{k-1}$ to a copy of $R^2 \otimes A_{k-1}$.

In its full glory, the map is given by:

$$\partial_k^{I \otimes A} : R^2 \otimes A_k \oplus R \otimes A_{k-1} \to R^2 \otimes A_{k-1} \oplus R \otimes A_{k-2}$$

$$\partial_k^{I \otimes A} = \begin{pmatrix} (c_1, c_2) \otimes a \\ c_3 \otimes \bar{a} \end{pmatrix} \mapsto \begin{pmatrix} (c_1, c_2) \otimes \partial a + (c_3, -c_3) \otimes \bar{a} \\ -c_3 \otimes \partial \bar{a} \end{pmatrix}$$

To begin the construction of the cylinder functor in a category of chain complexes, we should define what the functor does on objects and morphisms. We'll pick the assignments:

$$Cyl(A) :\equiv I \otimes A$$

$$(Cyl(f))_k(\begin{pmatrix} (c_1, c_2) \otimes a \\ c_3 \otimes \bar{a} \end{pmatrix}) :\equiv \begin{pmatrix} (c_1, c_2) \otimes f(a) \\ c_3 \otimes f(\bar{a}) \end{pmatrix}$$

It's easy to see that $Cyl(-)$ respects identities and composition, but since the second assignment was given explicitly, we must verify that it is actually a chain map. First, note that for fixed k, the above assignment does give an R-module homomorphism through a few applications of the universal properties of $\otimes$ and $\oplus$. Then, if we examine the diagram:

$$\begin{tikzcd} Cyl(A)_k \arrow[r, "Cyl(f)_k"] \arrow[d, "\partial^{I\otimes A}"'] & Cyl(B)_k \arrow[d, "\partial^{I\otimes B}"] \\ Cyl(A)_{k-1} \arrow[r, "Cyl(f)_{k-1}"] & Cyl(B)_{k-1} \end{tikzcd}$$

tracing the effect of the top and bottom paths on a generic element $\begin{pmatrix} (c_1, c_2) \otimes a \\ c_3 \otimes \bar{a} \end{pmatrix}$ of $Cyl(A)_k$ yields

$$\begin{pmatrix} (c_1, c_2) \otimes \partial f(a) + (c_3, -c_3) \otimes f(\bar{a}) \\ -c_3 \otimes \partial f(\bar{a}) \end{pmatrix}$$

and

$$\begin{pmatrix} (c_1, c_2) \otimes \partial f(a) + (c_3, -c_3) \otimes f(\bar{a}) \\ -c_3 \otimes f(\partial \bar{a}) \end{pmatrix}$$

respectively, and they agree since f is a chain map.

From there, we need to construct the projection maps $\sigma_A : Cyl(A) \to A$. Since for each component of this chain map, we'll need to construct R-module homomorphisms of type $R^2 \otimes A_k \oplus R \otimes A_{k-1} \to A_k$, and there's not (in general) a consistently-defined non-zero homomorphism $A_{k-1} \to A_k$, we should not use the second coordinate of the argument in the definition of σ_A. Then, to make the square:

$$\begin{tikzcd} Cyl(A)_k \arrow[r, "\sigma_{Ak}"] \arrow[d, "\partial^{I\otimes A}"'] & A_k \arrow[d, "\partial^{A}"] \\ Cyl(A)_{k-1} \arrow[r, "\sigma_{A(k-1)}"] & A_{k-1} \end{tikzcd}$$

commute, the appearance of a term of the form $(c_3, -c_3) \otimes \bar{a}$ in the first coordinate of the result of applying $\partial^{I\otimes A}$ essentially mandates the choice:

$$\sigma_{Ak}(\begin{pmatrix} (c_1, c_2) \otimes a \\ c_3 \otimes \bar{a} \end{pmatrix}) = (c_1 + c_2)a$$

to make the bottom path commute with the top, since the path on top kills off the second coordinate of its input. The claim that σ gives natural transformation from Cyl to Id is easily verified through routine calculations.

For the natural transformations d_i, noting first the condition $\sigma d_i = Id$, it's easy to see that each component $d_{iAk} : A_k \to R^2 \otimes A_k \oplus R \otimes A_{k-1}$ must send $a : A_k$ to something whose first coordinate is either $(1,0) \otimes a$ or $(0,1) \otimes a$. From there, the condition that each d_i is a chain map mandates that the second coordinate of $d_{iAk}(a)$ is always zero, because if not, in the square:

$$\begin{tikzcd} A_k \arrow[r, "d_{iAk}"] \arrow[d, "\partial^A"'] & Cyl(A)_k \arrow[d, "\partial^{I\otimes A}"] \\ A_{k-1} \arrow[r, "d_{iA(k-1)}"'] & Cyl(A)_{k-1} \end{tikzcd}$$

tracing the effect of the bottom path on $a : A_k$ would yield $(1,0) \otimes \partial^A a$ or $(0,1) \otimes \partial^A a$ for the first coordinate, but the top path would yield an additional term which scales linearly with $\pi_2(d_{iAk}(a))$, and so $\pi_2(d_{iAk}(a)) = 0$. As a result, these maps are uniquely fixed and given explicitly by:

$$d_{0Ak}(a) = \begin{pmatrix} (1,0) \otimes a \\ 0 \end{pmatrix}$$

$$d_{1Ak}(a) = \begin{pmatrix} (0,1) \otimes a \\ 0 \end{pmatrix}$$

The verification that each yields a natural transformation from Id to Cyl is routine.

This completes the construction of a cylinder functor on any category of chain complexes, and so we may now discuss homotopies in such a category.

1.5.3 Chain Homotopies and Homotopies between Chain Maps

Going back to the definition of a chain homotopy, we had that a chain homotopy between chain maps $f, g : A \to B$ was a collection of $h_n : A_n \to B_{n+1}$ such that $f - g = \partial h + h\partial$. In contrast, a homotopy between chain maps in a category of chain complexes is an η fitting into the commutative homotopy diagram:

$$\begin{tikzcd} A \arrow[d, "d_{0A}"'] \arrow[dr, "f"] & \\ Cyl(A) \arrow[r, "\eta"] & B \\ A \arrow[u, "d_{1A}"] \arrow[ur, "g"'] & \end{tikzcd}$$

The connection between these two concepts may be seen by examining the consequences of the commutativity of this diagram. Suppose that we have some $a : A_k$. Then, we need both:

$$\eta_k(\begin{pmatrix} (1,0) \otimes a \\ 0 \end{pmatrix}) = f_k(a)$$

$$\eta_k(\begin{pmatrix} (0,1) \otimes a \\ 0 \end{pmatrix}) = g_k(a)$$

and so by the universal properties of the tensor product and the direct sum, the response of η to the first coordinate is completely determined by f and g.

However, to completely specify η, we also need to specify a $\eta_k(\begin{pmatrix} 0 \\ c \otimes a \end{pmatrix}) = c\, h_{k-1}(a)$ by specifying a collection of R-module homomorphisms $h_k : A_k \to B_{k+1}$. This should already look suspiciously similar to the data for a chain homotopy.

The only constraint we haven't yet examined is the requirement that η is a chain map. If we chase the effect of the two paths in the chain map commutative diagram on a generic element $\begin{pmatrix} (c_1, c_2) \otimes a \\ c_3 \otimes \bar{a} \end{pmatrix}$ of $Cyl(A)_k$, we obtain the following:

$$\begin{tikzcd}
Cyl(A)_k \arrow[r, "\eta_k"] \arrow[d, "\partial^{I\otimes A}"'] & B_k \arrow[d, "\partial^B"] \\
Cyl(A)_{k-1} \arrow[r, "\eta_{k-1}"] & B_{k-1}
\end{tikzcd}$$

$$\begin{tikzcd}
\begin{pmatrix} (c_1, c_2) \otimes a \\ c_3 \otimes \bar{a} \end{pmatrix} \arrow[r] \arrow[dd] & c_1 f(a) + c_2 g(a) + c_3 h_{k-1}(\bar{a}) \arrow[d] \\
 & c_1 \partial f(a) + c_2 \partial g(a) + c_3 \partial h_{k-1}(\bar{a}) \arrow[d, "=?"] \\
\begin{pmatrix} (c_1, c_2) \otimes \partial a + (c_3, -c_3) \otimes \bar{a} \\ -c_3 \otimes \partial \bar{a} \end{pmatrix} \arrow[r] & f(c_1 \partial a + c_3 \bar{a}) + g(c_2 \partial a - c_3 \bar{a}) - c_3 h_{k-2}(\partial \bar{a})
\end{tikzcd}$$

From this, and applying the fact that f, g are chain maps, we can see that this diagram will commute if and only if

$$f_{k-1}(\bar{a}) - g_{k-1}(\bar{a}) - h_{k-2}(\partial \bar{a}) = \partial h_{k-1}(\bar{a})$$

which, after re-arranging, is exactly the chain homotopy condition

$$f - g = \partial h + h \partial$$

So, in general, if we have a chain homotopy between $f, g : X \to Y$, then it provides the necessary data to uniquely determine a homotopy in the category of chain complexes between f and g. Explicitly, if we have a chain homotopy h between f and g, we'll refer to the *induced homotopy* as the chain map $\eta : Cyl(X) \to Y$ given by

$$\eta_k(\begin{pmatrix} (c_1, c_2) \otimes a \\ c_3 \otimes \bar{a} \end{pmatrix}) = c_1 f_k(a) + c_2 g_k(a) + c_3 h_{k-1}(\bar{a})$$

2 Strong Deformation Retracts

Recall that a (strong) *deformation retract* from X onto $A \subseteq X$ in Top is a homotopy from id_X to a retraction onto A such that when the homotopy is sliced as a family of functions

$h_t : X \to A$, each $h_t|_A = id_A$. We can generalize this definition to the case of any category with a cylinder functor as follows:

A *strong deformation retract* from X onto A is given by a retraction $r : X \to A$ with section $i : A \to X$ ($r \circ i = Id_A$) such that there's a homotopy η from $i \circ r$ to Id_X fitting into the diagram:

$$\begin{tikzcd}
X \arrow[r, "r"] \arrow[d, "d_{0A}"'] & A \arrow[d, "i"] \\
Cyl(X) \arrow[r, "\eta"] & X \\
X \arrow[u, "d_{1A}"] \arrow[ur, "Id_X"']
\end{tikzcd}$$

such that the identity $\eta \circ Cyl(i) = \sigma_X \circ Cyl(i)$ holds (*).

Intuitively, the last condition expresses that the effect of η is constant on the cylinder over A mapped into X by i.

Now, to interpret this in a category of chain complexes, our previous notes on chain homotopies imply that to define such a strong deformation retract, we just need a chain homotopy $\{h_n : X_n \to X_{n+1} \,|n \in \mathbb{N}\}$ from $i \circ r$ to Id_X such that an appropriate translation of condition (*) holds.

By using the standard method to obtain η from the chain homotopy h, we can see that the left hand side of (*) applied to a generic element is:

$$\eta \circ Cyl(i)(\begin{pmatrix} (c_1, c_2) \otimes a \\ c_3 \otimes \bar{a} \end{pmatrix}) = \eta \begin{pmatrix} (c_1, c_2) \otimes i(a) \\ c_3 \otimes i(\bar{a}) \end{pmatrix} = c_1 i(r(i(a))) + c_2 i(a) + c_3 h_{k-1}(i(\bar{a}))$$

and the right-hand side applied to the same element is:

$$\sigma_X \circ Cyl(i)(\begin{pmatrix} (c_1, c_2) \otimes a \\ c_3 \otimes \bar{a} \end{pmatrix}) = (c_1 + c_2) i(a)$$

Using the fact that $r \circ i = Id_A$, for these two expressions to be equal, it's necessary and sufficient for $h \circ i$ to be the zero map.

So to summarize, a strong deformation retract from X onto A in the category of chain complexes is described by:

- $r : X \to A$, $i : A \to X$ chain maps
- $\{h_n : X_n \to X_{n+1} \,|n \in \mathbb{N}\}$ a collection of homomorphisms

Satisfying

- $r \circ i = Id_A$
- $h\partial^X + \partial^X h = i \circ r - Id_X$
- $h \circ i = \mathbf{0}$

To distinguish data satisfying these conditions from "SDR data" or "reductions" (which satisfy additional conditions), we'll call any such (X, A, r, i, h) *Essential SDR data*.

The most crucial feature of a (strong, or even weak) deformation retract of chain complexes is that if there's a deformation retract from X to A, then for every $n \in \mathbb{N}$, the n-th homology groups of each are isomorphic – $H_n(X) \simeq H_n(A)$. In more fancy terminology, a deformation retract induces a *quasi-isomorphism* between chain complexes. At a very high level, this is true because a strong deformation retract is a special kind of chain equivalence.

Seeing this explicitly is also easy, since we can simply notice that if $(-)_*$ denotes the homology functor,

$$r_* \circ i_* = (Id_A)_*$$

but also, since we know that

$$h\partial^X + \partial^X h = i \circ r - Id_X$$

if we apply the homology functor to both sides and track what would happen to a typical element, the $h\partial^X$ contribution kills off all cycles, and the remaining contribution due to $\partial^X h$ is a boundary term, and so the induced map on homology is zero. Hence,

$$i_* \circ r_* = (Id_X)_*$$

so i_* and r_* are mutual inverses, and so the two complexes have the same homology groups.

2.1 Additional Conditions

To massage this definition of "Essential SDR data" into SDR data, we can examine additional desirable conditions on a strong deformation retract.

2.1.1 Naturality of Homotopy

Classically, if we have a homotopy $H : I \times X \to X$ from f to id_X, it's necessarily the case that the diagram

$$\begin{tikzcd} I \times X \arrow[r, "H"] \arrow[d, "Id_I \times f"] & X \arrow[d, "f"] \\ I \times X \arrow[r, "H"] & X \end{tikzcd}$$

commutes up to a homotopy, because we can give a homotopy explicitly by $t_1 \mapsto t_2 \mapsto x \mapsto H_{t_1 * t_2}(H_{(1-t_1)*t_2}(x))$. It turns out that a proper analogue of this also holds in the particular case of our Strong Deformation Retracts above, namely that if η is the homotopy induced by an h as above:

$$\begin{tikzcd}
Cyl(X) \arrow[r, "\eta"] \arrow[d, "Cyl(ir)"] & X \arrow[d, "ir"] \\
Cyl(X) \arrow[r, "\eta"] & X
\end{tikzcd}$$

commutes up to homotopy. To prove this, we must provide a chain homotopy $\{H_n : Cyl(X)_n \to X_{n+1} \,|n \in \mathbb{N}\}$ such that

$$ir\eta - \eta Cyl(ir) = \partial^X H + H\partial^{I\otimes X}$$

To track the effect of the left-hand side on our favorite generic element $v = \begin{pmatrix} (c_1, c_2) \otimes a \\ c_3 \otimes \bar{a} \end{pmatrix}$ of $Cyl(X)_k$, note that

$$ir\eta(v) = ir(c_1 ir(a) + c_2 a + c_3 h_{k-1}(\bar{a}))$$

$$= c_1 i(ri)r(a) + c_2 ir(a) + c_3 irh_{k-1}(\bar{a})$$
$$= (c_1 + c_2)ir(a) + c_3 irh_{k-1}(\bar{a})$$

and

$$(\eta \circ Cyl(ir))(v) = c_1 irir(a) + c_2 ir(a) + c_3 h_{k-1}(ir(\bar{a}))$$

So in total, the left-hand side yields:

$$(ir\eta - \eta Cyl(ir))(v) = c_3(irh_{k-1}(\bar{a}) - h_{k-1}ir(\bar{a}))$$

For our homotopy H, pick the map:

$$H(\begin{pmatrix} (c_1, c_2) \otimes a \\ c_3 \otimes \bar{a} \end{pmatrix}) = h(h(\bar{a}))$$

Then, applying the boundary map $\partial^{I\otimes X}$ first and then H yields a contribution $-c_3 h(h(\partial^X \bar{a}))$, and doing it in the opposite order yields a contribution $c_3 \partial^X h(h(\bar{a}))$. So we only need to demonstrate that

$$irh_{k-1}(\bar{a}) - h_{k-1}ir(\bar{a}) = \partial^X h(h(\bar{a})) - h(h(\partial^X \bar{a}))$$

but since we know that $h\partial^X + \partial^X h + Id_X = i \circ r$, the LHS becomes:

$$(h\partial^X h + \partial^X hh + h - (hh\partial^X + h\partial^X h + h))(\bar{a})$$

$$= (\partial^X hh - hh\partial^X)(\bar{a})$$

and so H is indeed a homotopy expressing commutativity of the square.

2.1.2 Forcing Strict Commutativity of the Naturality Diagram

While the argument above shows that the homotopy naturality square commutes up to homotopy, suppose that we wanted to force the diagram to simply *commute* (not up to some higher homotopy). This means that we would need to force

$$ir\eta - \eta Cyl(ir) = 0$$

but from the computations above, we can see that this is equivalent to forcing

$$irh = hir$$

Now, since $h \circ i = 0$ for a strong deformation retract, we only need to force $irh = 0$, so it would work to impose the extra condition

- $r \circ h = \mathbf{0}$

Given essential SDR data, it's easy to slightly modify the homotopy h to obtain a homotopy $\bar{h}$ satisfy this condition and all of the others. Let

$$\bar{h} = h - irh$$

Then, note that

$$\bar{h}\partial^X + \partial^X \bar{h} = h\partial^X + \partial^X h - irh\partial^X - \partial^X irh$$

$$= i \circ r - Id_X - ir(h\partial^X + \partial^X h)$$
$$= i \circ r - Id_X - ir(ir - Id_X) = i \circ r - Id_X$$

and so $\bar{h}$ is still a homotopy from $i \circ r$ to the identity, but also

$$\bar{h} \circ i = hi - hiri = \mathbf{0}$$

and so the deformation retract is still strong. Finally, we can compute:

$$r\bar{h} = rh - rirh = \mathbf{0}$$

So indeed, this transformation forces the naturality diagram to commute strictly.

2.1.3 Overkill: Killing off the Non-trivial Null Homotopy Generator

After imposing the condition $r \circ h = \mathbf{0}$, we actually find ourselves in a case where there are two distinct homotopies expressing commutativity of the naturality diagram – since the diagram commutes strictly, we could pick our old homotopy induced by H, but we could also pick the homotopy induced by the chain homotopy $\mathbf{0}$. However, since H is not necessarily zero, it would be nice to kill off this extra generator in the space of the homotopies expressing commutativity of the diagram by imposing an extra constraint. Recalling the definition of H, we can see that forcing:

- $h \circ h = \mathbf{0}$

is exactly what we need to make this happen.

Now, suppose that we have essential SDR data which satisfies the additional condition $r \circ h = \mathbf{0}$. We can construct a new collection of maps:

$$\bar{h} = -h\partial^X h$$

which clearly satisfies $r \circ \bar{h} = \mathbf{0}$ and $\bar{h} \circ i = \mathbf{0}$. We need to show that it's still a chain homotopy. First, note that we can rewrite $\bar{h}$ in two different ways using the chain homotopy condition on h:

$$\bar{h} = -h(ir - Id - h\partial^X) = h + hh\partial^X$$

$$\bar{h} = -(ir - Id - \partial^X h)h = h + \partial^X hh$$

One additional consequence of this rewriting is that $hh\partial^X = \partial^X hh$, that is, $h \circ h$ is a chain map from X to $\Sigma^2 X$. Then,

$$\bar{h}\partial^X + \partial^X \bar{h} = (h + hh\partial^X)\partial^X + \partial^X(h + \partial^X hh)$$

$$= h\partial^X + \partial^X h$$

and so all of the previous conditions are satisfied. Then, note that

$$\bar{h}\bar{h} = h\partial^X(hh\partial^X)h = h(\partial^X\partial^X)hhh = \mathbf{0}$$

as desired.

3 Constructive Algebraic Topology: Setup

Constructive Algebraic Topology is entirely based on the idea of relating a chain complex which does not necessarily admit a finite description in each dimension to a chain complex which does through a strong deformation retract. In this way, the homology groups of the "infinite" complex may be computed by performing elementary linear algebra on the finite one. An chain complex which has finitely-many generators and computable boundary maps in every dimension is said to be *effective*, but a chain complex which has computable boundary maps such that the generators in each dimension are *computably enumerable* is said to be only *locally effective*.

Now, some definitions: A *reduction* ρ (or a collection of "SDR data") from X to A, denoted $X \overset{\rho}{\twoheadrightarrow} A$ is a strong deformation retract as described earlier, satisfying all extra conditions considered above. That is, it's a tuple (X, A, i, r, h) satisfying:

- $r : X \to A$, $i : A \to X$ chain maps
- $\{h_n : X_n \to X_{n+1} \,|n \in \mathbb{N}\}$ a collection of homomorphisms
- $r \circ i = Id_A$

- $h\partial^X + \partial^X h = Id_X - i \circ r$
- $h \circ i = \mathbf{0}$
- $h \circ h = \mathbf{0}$
- $r \circ h = \mathbf{0}$

Here, we have swapped the direction of the homotopy h (which is now from the identity to $i \circ r$) to stick with the common convention adopted by resources on homological perturbation theory. While it's somewhat more intuitive to think about "homotoping $i \circ r$ away," the convention somewhat simplifies book-keeping in the proof of the Basic Perturbation Lemma.

Then, a *homotopy equivalence* between X and Y is another chain complex Z together with a pair of reductions:

$$\begin{tikzcd} & Z \arrow[dl, "\rho_1"'] \arrow[dr, "\rho_2"] & \\ X & & Y \end{tikzcd}$$

So if there's a homotopy equivalence between X and Y, all of their homology groups agree. Z doesn't need to be effective, and may be locally effective.

Finally, an *object with effective homology* consists of a locally effective complex C, an effective complex EC and a homotopy equivalence between them.

Then, it's the task of constructive algebraic topology is to find ways to construct interesting and useful examples of objects with effective homology. Every effective complex may be trivially transformed into an object with effective homology, but interestingly enough, there are many examples of only locally-effective complexes which nevertheless have corresponding objects with effective homology. One example of this is the chain complex on the infinite-dimensional simplex Δ^∞, which is only locally effective in every dimension, but also is homotopy equivalent to the zero chain complex.

4 The Swiss Army Knife of CAT: The Basic Perturbation Lemma

The fundamental idea behind CAT's applications of the Basic Perturbation Lemma is that we can mess with the differential on the top complex Z in an object with effective homology and then propagate these changes down to X and Y through the reductions to produce a new object with effective homology.

First, a *perturbation* of the differential ∂^Z on Z is a collection of homomorphisms $\{\delta_n : Z_{n+1} \to Z_n\}_{n\in\mathbb{N}}$ such that $\partial^Z + \delta$ defines a differential on Z, or explicitly,

$$(\partial^Z + \delta) \circ (\partial^Z + \delta) = 0$$

.

It's trivial but very important to note that if we have two different differentials ∂^Z and $\bar{\partial}^Z$ on Z, then the difference $\bar{\partial}^Z - \partial^Z$ is a perturbation.

In the situation where we have a reduction $Z \overset{(i,r,h)}{\twoheadrightarrow} X$, a perturbation δ of ∂^Z is *locally nilpotent* if for every $z : Z_m$ (for arbitrary m), there's some $n \in \mathbb{N}$ such that $(h \circ \delta)^n(z) = 0$

Then, the statement of the Basic Perturbation Lemma is simply that in this situation, we can construct a new reduction $\bar{Z} \overset{(\bar{i},\bar{r},\bar{h})}{\twoheadrightarrow} \bar{X}$ where $\bar{Z}$ is Z but with ∂^Z replaced with $\partial^Z + \delta$ and $\bar{X}$ is X but with ∂^X replaced with $\partial^X + \bar{\delta}$ for some perturbation $\bar{\delta}$. Or, using a cleaner notation for this situation which includes differentials with all complexes,

$$(Z, \partial^Z + \delta) \overset{(\bar{i},\bar{r},\bar{h})}{\twoheadrightarrow} (X, \partial^X + \bar{\delta})$$

Remark: Note that if we considered the challenge of propagating a perturbation of a differential *the other way* in a reduction (so that we use a perturbation on the differential on the *target* of a SDR to obtain a perturbation on the *source*), we do not need the sophisticated machinery of the BPL. In particular, if we have $Z \overset{(i,r,h)}{\twoheadrightarrow} X$ and *any* perturbation δ^X, using the conditions on the original SDR it's straightforward to verify that there is a new reduction:

$$(Z, \partial^Z + i\delta^X r) \overset{(i,r,h)}{\twoheadrightarrow} (X, \partial^X + \delta^X)$$

4.1 Sample of Usage: The Homology Groups of a Product of Spaces and Bicomplexes

From an earlier remark, we know that the ordinary homology groups of the space $X \times Y$ are the same as the homology groups of $T(Ch(X) \otimes Ch(Y))$. Supposing that both $Ch(X)$ and $Ch(Y)$ have corresponding objects with effective homology O_1 and O_2, the new goal is to construct an object with effective homology $O_1 \otimes O_2$ which contains a homotopy equivalence between $Ch(X) \otimes Ch(Y)$ and some effective complex.

Here, we'll simplify matters a little by supposing that one of our objects (which we will simply refer to as X) is already effective, meaning that all chain complexes in the definition of an object with effective homology are the same, and all of the reductions are trivial.

Note: We can actually do this by constructing the *tensor product of reductions*, which will actually work in more general cases than the one considered above, but this is perhaps the most elementary example of something obeying the hypotheses required to compute an object with effective homology of the total complex of a bicomplex. The (incredibly intuitive) algorithm for computing the total complex of a bicomplex also demonstrates how the basic perturbation lemma can be used to replace certain elementary examples of spectral sequences.

First, expanding the definition of an object with effective homology, we have reductions:

$$\begin{tikzcd}
 & Y \arrow[dl, two heads, "\rho_L"'] \arrow[dr, two heads, "\rho_E"] & \\
L_Y & & E_Y
\end{tikzcd}$$

Which we will write out in components as:

$$\begin{tikzcd}
Y \arrow[loop left, "\partial^Y"] \arrow[loop right, "h^{L_Y}"] \arrow[d, bend right, "r^{L_Y}"'] \\
L_Y \arrow[loop left, "\partial^{L_Y}"] \arrow[u, bend right, "i^{L_Y}"']
\end{tikzcd}
\qquad
\begin{tikzcd}
Y \arrow[loop left, "\partial^Y"] \arrow[loop right, "h^{E_Y}"] \arrow[d, bend right, "r^{E_Y}"'] \\
E_Y \arrow[loop left, "\partial^{E_Y}"] \arrow[u, bend right, "i^{E_Y}"']
\end{tikzcd}$$

Where L_Y is a locally-effective complex and E_Y is an effective one.

In this setting, we'll first note that using the diagram from 1.4, we can see that the columns of the tensor product double complex are chain complexes where the R-modules are of the form $\{X_k \otimes Y_m\}_{m\in\mathbb{N}}$ for some fixed k and the differential is given by $1 \otimes \partial^Y$. Denote this complex by $X_k \otimes Y$. From this, we may obtain reductions for every $k \in \mathbb{N}$ of the forms:

$$\left\{
\begin{tikzcd}
X_k \otimes Y \arrow[loop left, "1\otimes\partial^Y"] \arrow[loop right, "1\otimes h^{E_Y}"] \arrow[d, bend right, "1\otimes r^{E_Y}"'] \\
X_k \otimes E_Y \arrow[loop below, "1\otimes\partial^{E_Y}"] \arrow[u, bend right, "1\otimes i^{E_Y}"']
\end{tikzcd}
\right\}_{k\in\mathbb{N}}
\left\{
\begin{tikzcd}
X_k \otimes Y \arrow[loop left, "1\otimes\partial^Y"] \arrow[loop right, "1\otimes h^{L_Y}"] \arrow[d, bend right, "1\otimes r^{L_Y}"'] \\
X_k \otimes L_Y \arrow[loop below, "1\otimes\partial^{L_Y}"] \arrow[u, bend right, "1\otimes i^{L_Y}"']
\end{tikzcd}
\right\}_{k\in\mathbb{N}}$$

Here, wherever we display a tensor product of maps, e.g. $1 \otimes i^Y$, we mean that we are taking a pointwise tensor product of the right-hand map on every degree of a graded R-module with the left-hand R-module homomorphism.

In other words, we have a reduction for every column in the tensor product double complex. We would like to be able to compute an object with effective homology which provides a homotopy equivalence between $T(X \otimes L_Y)$ and some effective complex. Luckily, there's an easy way to do this: On the left-hand side of the reduction, recall that as a graded module, the tensor product is given in each degree:

$$\overline{X \otimes L_Y}_n = \oplus_{i+j=n} X_i \otimes Y_j$$

Now, by simply *ignoring* the horizontal differentials in the tensor product diagram, we can get a differential $\bar{\partial}_n^{\overline{X\otimes L_Y}}$ defined on the direct summands by:

$$\bar{\partial}_n^{\overline{X\otimes L_Y}}(x \otimes y) :\equiv (-1)^{deg(x)} x \otimes (\partial^{L_Y} y) = (-1)^{deg(x)} \partial^V (x \otimes y)$$

and we may also define analogous differentials on $\overline{X \otimes Y}$ and $\overline{X \otimes E_Y}$. Here and in the following, we use the overline to emphasize that for given choices of Z, we are only claiming that $\overline{X \otimes Z}$ has the same structure as $X \otimes Z$ *as a graded module*.

Then, it's clear that we obtain a reduction:

$$\begin{tikzcd}
\overline{X \otimes Y} \arrow[loop left, "\bar{\partial}^{\overline{X\otimes Y}}"] \arrow[loop right, "\oplus(1\otimes h^Y)"] \arrow[d, bend right, "\oplus(1\otimes r^{L_Y})"'] \\
\overline{X \otimes L_Y} \arrow[u, bend right, "\oplus(1\otimes i^{L_Y})"'] \arrow[loop below, "\bar{\partial}^{\overline{X\otimes L_Y}}"]
\end{tikzcd}$$

where expressions like "$\oplus(1 \otimes i^{L_Y})$" denote (suitably indexed and delimited) direct sums of maps used in the column reductions. We'll call this reduction $1 \otimes \rho_L$. We may also do this for E_Y to obtain a new homotopy equivalence:

$$\begin{tikzcd}
& (\overline{X \otimes Y}, \bar{\partial}^{\overline{X\otimes Y}}) \arrow[dl, "1\otimes\rho_L"'] \arrow[dr, "1\otimes\rho_E"] & \\
(\overline{X \otimes L_Y}, \bar{\partial}^{\overline{X\otimes L_Y}}) & & (\overline{X \otimes E_Y}, \bar{\partial}^{\overline{X\otimes E_Y}})
\end{tikzcd}$$

Where the overline is used above to emphasize the fact that the differentials here prevent us from truly calling these things tensor products. We would like to perturb the differential on the left-hand complex to fix this situation.

Note that we can define another differential $\partial^{\overline{X\otimes L_Y}}$ on the left-hand complex given by the *horizontal* differentials:

$$\partial_n^{\overline{X\otimes L_Y}}(x \otimes y) :\equiv (\partial^X x) \otimes y = \partial^H(x \otimes y)$$

Then, we know that the true differential on the tensor product $\partial^{X\otimes L_Y}$ is $\bar{\partial}^{\overline{X\otimes L_Y}} + \partial^{\overline{X\otimes L_Y}}$. So we can consider this second differential as a perturbation of the first differential, and using the remark of the previous section, we can use this to obtain a reduction between the perturbed top complex and the perturbed left complex, which is now $X \otimes L_Y$. All that remains is to propagate the perturbation

$$\delta :\equiv (\oplus 1 \otimes i^{L_Y}) \circ \partial^{\overline{X\otimes L_Y}} \circ (\oplus 1 \otimes r^{L_Y})$$

on

$$(\overline{X \otimes Y}, \bar{\partial}^{\overline{X\otimes Y}})$$

down to

$$(\overline{X \otimes E_Y}, \bar{\partial}^{\overline{X\otimes E_Y}})$$

, whence we will finally have a homotopy equivalence between $X \otimes L_Y$ and some effective chain complex.

To do so, the basic perturbation lemma only requires that we show that the perturbation on the top complex is locally nilpotent. However, this is easy – the homotopy operator on the top complex is given by $h :\equiv (\oplus 1 \otimes h^Y)$, which is a direct sum of homotopies which increase the dimension by going one step *up* a *column*, but the perturbation was given by the horizontal differentials, which go one step *down* a *row*. As a result, for any x, $(h \circ \delta)^n(x) = 0$, since we can simply take n to be the highest horizontal degree in the components of x. So indeed, we give a structure on $X \otimes L_Y$ as an object with effective homology.

Note that in the above, there was absolutely nothing special about the fact that we considered the bicomplex corresponding to a tensor product of chain complexes: the above derivation works in the general case of bicomplexes where every column has an object with effective homology.

5 A Traditional (and Moral) Proof of the Basic Perturbation Lemma

The usual proof of the BPL ultimately comes down to verifying that if we set:

- $\psi = \sum_{i=0}^{\infty}(-1)^{(i)}(\delta h)^{(i)}$
- $\phi = \sum_{i=0}^{\infty}(-1)^{(i)}(h\delta)^{(i)}$
- $A = \psi\delta = \delta\phi$
- $\bar{h} = h\psi = \phi h$
- $\bar{i} = \phi i$
- $\bar{r} = r\psi$
- $\bar{\delta} = rAi$

Note that all of the quantities above are well-defined *precisely* because of the condition that the perturbation is locally nilpotent, since this forces the operators ϕ and ψ to only incorporate finitely-many summands on any particular input.

Then all of the conditions on a strong deformation retract are satisfied for

$$(Z, \partial^Z + \delta) \xrightarrow{(\bar{i},\bar{r},\bar{h})} (X, \partial^X + \bar{\delta}\ )$$

, which we may verify after some careful computation. For convenience, we'll define

$$\bar{\partial} \equiv \partial^Z + \delta$$

$$\bar{\partial}^X \equiv \partial^X + \bar{\delta}$$

and omit the superscript on ∂^Z whenever it appears.

However, it's also important to understand why this is true *morally* – we could have pulled those assignments outta nowhere! In particular, it would be nice to know what choices (if any) force these definitions upon us. We will examine this before verifying that the above assignments work.

5.1 The Morality of the BPL

Starting from square one, we know that in our resulting reduction, we'll need some kind of chain homotopy $\bar{h}$ between $\bar{i}\bar{r}$ and the identity. Now, since chain homotopies are operators on a chain complex which boost dimension by 1, $\bar{h}$ must do so on Z. Looking through our repertoire of functions, we only have a single operator, h, which boosts dimensions. So we could optimistically set $\bar{h} = h\psi$ for some (currently) arbitrary dimension-preserving transform ψ.

However, noting that we need to also somehow force $\bar{h}^2 = 0$, we seem stuck with the expression $h\psi h\psi$. It would be nice to be able to pull h past the ψ to use $h^2 = 0$, so we also postulate that there's a dimension-preserving ϕ such that $\phi h = \psi h$. As consequences, we have

$$\bar{h}^2 = \bar{h}h = h\bar{h} = h^2 = 0$$

From those humble beginnings, all of the other formulas above will follow naturally. First, we need to introduce a different perspective.

5.1.1 Projections and Splitting

Recall that a projector is a linear operator P such that $P^2 = P$. We have the following:

Lemma. If we have linear operators h and ∂ on Z such that $h^2 = 0 = \partial^2$ and $h\partial h = h$, then

$$P = h\partial + \partial h$$

is a projector.

Proof.

$$P^2 = (h\partial + \partial h)(h\partial + \partial h) = h\partial h\partial + \partial h\partial h = h\partial + \partial h = P$$

□

Recalling what we know about linear projectors, we know that

$$ker(P) \simeq Im(1 - P)$$

and also that

$$Z \simeq ker(P) \oplus Im(P) \simeq Im(1 - P) \oplus Im(P)$$

Now, in the setting of the BPL, this result applies for π set as above, since

$$h\partial h = (1 - ir - \partial h)h = h$$

Hence, we know that

$$Z \simeq Im(ir) \oplus Im(\partial h + h\partial)$$

However, we know that as a chain complex, all of the homology groups of $Im(\partial h + h\partial)$ are zero! So any essential information about the retracted complex X is contained in the $Im(ir)$ component of Z.

Now, in the perturbed complex, assuming that we demonstrate that $\bar{h}$ is a homotopy between $\bar{i}\bar{r}$ and the identity and that $\bar{r} \circ \bar{h} = 0$, we'll also be able to apply the lemma above in this setting to get:

$$\bar{P} = \bar{\partial}\bar{h} + \bar{h}\bar{\partial}$$

$$\bar{Z} \simeq Im(\bar{i}\bar{r}) \oplus Im(\bar{\partial}\bar{h} + \bar{h}\bar{\partial})$$

We would like to somehow draw a connection between the summands that really matter in Z and $\bar{Z}$: $Im(ir)$ and $Im(\bar{i}\bar{r})$. For now, we'll just leave off with the definitions:

$$\pi = 1 - P \quad (= ir)$$

$$\bar{\pi} = 1 - \bar{P} \quad (= \bar{i}\bar{r})$$

and note that we need a connection between $Im(\pi)$ and $Im(\bar{\pi})$.

5.1.2 Pinning Down ψ and ϕ

First, we want:

$$\bar{h}\bar{i} = 0$$

But this would also imply that:

$$\bar{h}\bar{i}\bar{r} = 0$$

and keeping in mind that we also want $\bar{h}$ to satisfy $\bar{\partial}\bar{h} + \bar{h}\bar{\partial} = 1 - \bar{i}\bar{r}$, we need:

$$\bar{h}(1 - \bar{\partial}\bar{h} - \bar{h}\bar{\partial}) = 0$$

or (using $\bar{h}^2 = 0$)

$$\bar{h}(\partial + \delta)\bar{h} - \bar{h} = 0$$

but this is equivalent to:

$$\phi h\partial h\psi + h\psi\delta h\psi - h\psi = 0$$

$$= \phi h\psi + h\psi\delta h\psi - h\psi = 0$$
$$= \bar{h}((1 + \delta h)\psi - 1) = 0$$

An easy way to satisfy this derived condition would be to force

$$(1 + \delta h)\psi = 1$$

Once again, with enough cognitive impairment, we could interpret this as

$$\frac{1}{1 + \delta h} = \psi$$

and then use the typical geometric series expansion to arrive at:

$$\psi = \sum_{i=0}^{\infty} (-1)^{(i)} (\delta h)^{(i)}$$

By inspection of the formula $h\psi = \phi h$, it's clear that we also need:

$$\phi = \sum_{i=0}^{\infty} (-1)^{(i)} (h\delta)^{(i)}$$

Now, here are some useful identities for later:

$$A \equiv \psi\delta = \delta\phi \tag{1}$$
$$\delta h\psi = Ah = \psi\delta h = 1 - \psi \tag{2}$$
$$h\psi = \phi h \tag{3}$$
$$h\delta\phi = hA = \phi h\delta = 1 - \phi \tag{4}$$
$$\psi\phi = 1 - \psi - \phi \tag{5}$$
$$\tag{6}$$

5.1.3 Et Cetera

Now armed with a concrete definition for $\bar{h}$, we can look back at what we said about projectors to derive an isomorphism between $Im(\pi)$ and $Im(\bar{\pi})$. Note first that

$$\bar{h}\pi = h\pi = 0 = \pi\bar{h} = \pi h = (1 - h\partial - \partial h)h = h - h\partial h$$

$$\bar{\pi}h = \bar{\pi}\bar{h} = 0 = \bar{h}\bar{\pi} = h\bar{\pi} = h(1 - \bar{h}\bar{\partial} - \bar{\partial}\bar{h}) = h - h\partial h\psi - h\delta h\psi = h - h\psi - h(1 - \psi)$$

Then, it's easy to see that

$$(\bar{\pi}\pi)\bar{\pi} = \bar{\pi}(1 - h\partial - \partial h)\bar{\pi} = \bar{\pi}$$

$$(\pi\bar{\pi})\pi = \pi$$

In other words, $\bar{\pi}$ and π are mutual inverses as maps between $Im(\pi)$ and $Im(\bar{\pi})$.

However, $\pi\bar{\pi}$ and $\bar{\pi}\pi$ are *not* the identity. In fact,

$$\pi\bar{\pi} = \pi(1 - \bar{\partial}\bar{h}) = ir - ir\bar{\partial}\bar{h} = ir - ir\delta h\psi = \pi(1 - \delta h\psi) = \pi\psi$$

$$\bar{\pi}\pi = (1 - \bar{h}\bar{\partial})\pi = \phi\pi$$

Pinning Down $\bar{i}$: Now, suppose that we have any old map $f : Y \to Im(\pi)$ into $Im(\pi)$, and we want to transport along this identification between $Im(\pi)$ and $Im(\bar{\pi})$ to obtain a new map $\bar{f} : Y \to Im(\bar{\pi})$. Then, it's clear that

$$\bar{f} = \bar{\pi} f$$

From this, we can immediately motivate a definition for $\bar{i} : \bar{X} \to \bar{Z}$ as:

$$\bar{i} = \bar{\pi} i = (1 - \bar{\partial}\bar{h} - \bar{h}\bar{\partial})i = i - \bar{h}\bar{\partial}i = i - \bar{h}\delta i = (1 - \phi h \delta)i$$

$$= \phi i$$

Pinning Down $\bar{r}$: We want to have $\bar{\pi} = \bar{i}\bar{r} = \phi i \bar{r}$ (by homotopy) so left-multiplying by r and noting that $r\phi = r(1 - h\delta\phi) = r$ we get:

$$r\bar{\pi} = ri\bar{r} = \bar{r}$$

So

$$\bar{r} = r\bar{\pi} = r(1 - \bar{\partial}\bar{h}) = r(1 - \delta h \psi)$$

$$= r\psi$$

Pinning Down $\bar{\delta}$: Finally, in light of the isomorphism described by the projectors above, note that we want:

$$\bar{\partial}^X = \partial^X + \bar{\delta} = r\pi\bar{\partial}\bar{\pi}i$$

$$= rir\bar{\partial}\bar{\pi}i = \partial^X r\bar{\pi}i + r\delta\bar{\pi}i = \partial^X + r\delta\psi i$$

$$= \partial^X + rAi$$

So $\bar{\delta} = rAi$

5.2 A Thoroughly Algebraic Proof of the BPL

From the very definitions and some work done in the "Morality" section above, we can already verify:

$$\bar{h}^2 = \phi h^2 \psi = 0$$

$$\bar{r}\bar{h} = r\bar{\pi}\bar{h} = 0$$

$$\bar{h}\bar{i} = \bar{h}\bar{\pi}i = 0$$

$$\bar{r}\bar{i} = r\bar{\pi}\bar{\pi}i = r\bar{\pi}i = r(1 - \bar{\partial}\bar{h} - \bar{h}\bar{\partial})i = Id$$

But we still need to show that $\bar{\delta}$ is a perturbation of ∂^X, $\bar{h}$ gives a chain homotopy, and that $\bar{i}$, $\bar{r}$ are chain maps.

First, one thing that will be useful to us is that:

$$0 = \psi\bar{\partial}\bar{\partial}\phi$$
$$= \psi\partial\delta\phi + \psi\delta\partial\phi + \psi\delta^2\phi$$
$$= \psi\partial A + A\partial\phi + A^2$$

Or equivalently,

$$AirA + A\partial + \partial A = 0$$

since

$$A^2 - A\partial hA - Ah\partial A + A\partial + \partial A = A^2 + A\partial(1 - hA) + (1 - Ah)\partial A = \psi\partial A + A\partial\phi + A^2$$

We'll frequently use this in the form:

$$AirA = -(A\partial + \partial A)$$

$\bar{\delta}$ is a perturbation:

$$(\partial^X + \bar{\delta})^2 = (\partial^X + rAi)^2 = rAirAi + \partial^X rAi + rAi\partial^X = r(AirA + \partial A + A\partial)i = 0$$

$\bar{i}$ is a chain map: We need to show:

$$\bar{\partial}\bar{i} = \bar{i}(\partial^X + rAi)$$

But the RHS is:

$$\phi i(\partial^X + rAi) = (1 - hA)i(\partial^X + rAi) = i\partial^X - hAi\partial^X + irAi - h(AirA)i$$
$$= i\partial^X - h(A\partial)i + irAi + h(A\partial)i + h(\partial A)i$$
$$= i\partial^X + irAi + h(\partial A)i = \partial i + (1 - h\partial - \partial h)Ai + h(\partial A)i = \partial i + Ai - \partial hAi = (\partial + A - \partial hA)i$$
$$= (\partial(A - hA) + A)i = (\partial\phi + \delta\phi)i = \bar{\partial}\bar{i}$$

$\bar{h}$ is a chain homotopy: We have that:

$$1 - \bar{i}\bar{r} = 1 - \phi ir\psi = 1 - (1 - hA)ir(1 - Ah)$$
$$= 1 - (ir - hAir - irAh + h(AirA)h$$
$$= 1 - ir + (hA(ir) + (ir)hA + h\partial Ah + hA\partial h$$
$$= \partial h + h\partial + hA(ir + \partial h) + (ir + h\partial)Ah$$
$$= \partial h + h\partial + hA(1 - h\partial) + (1 - \partial h)Ah$$
$$= \partial h(1 - Ah) + (1 - hA)h\partial + Ah + hA$$
$$= \partial\bar{h} + \bar{h}\partial + \delta\bar{h} + \bar{h}\delta$$
$$= \bar{\partial}\bar{h} + \bar{h}\bar{\partial}$$

$\bar{r}$ **is a chain map:** We need to show:

$$\bar{r}\bar{\partial} = \bar{\partial}^X \bar{r}$$

But since we know that $\bar{i}$ is injective ($\bar{r}\bar{i} = 1$), it suffices to show:

$$\bar{i}\bar{r}\bar{\partial} = \bar{i}\bar{\partial}^X \bar{r} = \bar{\partial}^X \bar{i}\bar{r}$$

But from the "$\bar{h}$ is a chain homotopy" result above, since we know that 1 and $\bar{\partial}\bar{h} + \bar{h}\bar{\partial}$ are chain maps, so is $\bar{i}\bar{r}$, so the above equality holds.

So the Basic Perturbation Lemma has been proved.

6 Mechanizing an Elegant Proof of the BPL in HoTT

Now, the ending of the proof above may have seemed to require an excessive amount of work in verifying that all of the conditions for a strong deformation retract hold. Fundamentally, the argument above simply exploits the isomorphisms (as R-modules) between $Im(\bar{\pi})$ and $Im(\pi)$ and $Im(\pi)$ and A to obtain $\bar{\partial}^A$, $\bar{i}$ and $\bar{r}$ from a reduction ρ going from $\bar{X}$ to $Im(\bar{\pi})$. The following figures illustrate the situation:

$$\begin{tikzcd}
\bar{X} \arrow[d, "\rho"] \arrow[rrrr, dashed, "\bar{\rho}"] & & & & \\
Im(\bar{\pi}) \arrow[rr, equal, "P"'] & & Im(\pi) \arrow[rr, equal, "Q"'] & & \bar{A}(=A)
\end{tikzcd}$$

where P is an R-module isomorphism induced by π and $\bar{\pi}$, Q is another R-module isomorphism induced by i and r, and Here, we write $\bar{A}(=A)$ to emphasize the fact that the *reduction* $\bar{\rho}$ obtained by transporting ρ along the R-module isomorphisms along the bottom targets a copy of A with a perturbed differential ($\bar{A}$), but the bottom row is only concerned with *R-modules*, and from that perspective, $\bar{A}$ and A are identical.

Here, we can note that our remarks from 5.1.1 give us the reduction ρ by:

$$\begin{tikzcd}
\bar{X} \arrow[loop left, "\bar{\partial}"] \arrow[loop right, "\bar{h}"] \arrow[d, bend right, "\bar{\pi}"'] & \Longrightarrow & \bar{X} \arrow[d, "\rho"] \\
Im(\bar{\pi}) \arrow[u, bend right, "\mathbf{1}"'] & & Im(\bar{\pi})
\end{tikzcd}$$

for which we can check all of the conditions on SDR data, knowing already that $\bar{h}^2 = 0$ and that 1 and $\bar{\pi}$ are chain maps:

- $\bar{\pi} \circ 1 = \bar{\pi} = Id_{Im(\bar{\pi})}$ (since $\bar{\pi}$ is a projector)
- $Id_{\bar{X}} - \pi = Id_{\bar{X}} - (Id_{\bar{X}} - \bar{h}\bar{\partial} - \bar{\partial}\bar{h}) = \bar{h}\bar{\partial} + \bar{\partial}\bar{h}$

- $\bar{h} \circ 1 = 0$ (since $Im(\bar{\pi})$ is $Im(Id_{\bar{X}} - \bar{h}\bar{\partial} - \bar{\partial}\bar{h})$ and $\bar{h} - \bar{h}\bar{\partial}\bar{h} = 0$)
- $\bar{\pi} \circ \bar{h} = \bar{h} - \bar{h}\bar{\partial}\bar{h} = 0$

where in all of the above, we have written 1 for the inclusion of $Im(\bar{\pi})$ into $\bar{X}$ for brevity.

Now, traditionally, we could decide to prove the Basic Perturbation Lemma using this method, so long as we prove that transporting SDR data along an R-module isomorphism on the target of the reduction yields another reduction. Of course, the proof of this fact is entirely straightforward, and many authors omit the argument entirely in their informal exposition (such as Sergeraert in Cat.pdf). However, in a proof assistant, encoding this fact and its proof seems to be a necessary evil. In fact, this was the route taken for the formalization of the BPL in Isabelle/HOL as seen in ”A Mechanized Proof of the Basic Perturbation Lemma by Jess Aransay Clemens Ballarin Julio Rubio”.

Luckily, HoTT can help us get around needing to do this by an application of the Univalence Axiom: instead of manually mucking around with isomorphisms, we can write down a type family of reductions indexed by the target of the reduction (as an R-module). Then, referencing the diagram above, we can prove the BPL by transporting ρ in this type family over the composite equivalence $P \cdot Q$ where P and Q are derived by univalence. After we have done this, we may choose to verify that our usual formulas for the components of the reduction hold. However, this turns out to not be too terribly important: In our intended application of the BPL, we only care about ensuring that if we have an *effective* chain complex as the target of our initial reduction, then the BPL machine spits out a reduction to another effective chain complex. As a result, we only care about ensuring that the differential on the target remains computable, but this is easy, since (conjecturally) all expressions in HoTT are computationally realizable.

6.1 Preliminary Definitions

First, we note that definitions of Abelian groups and [augmented] (co)chain complexes have already been formalized in the HoTT-Agda library. While our formalization will make use of this library, we will make the following simplifications and modifications:

- In Agda, the types of all Abelian groups and all chain complexes are actually type families indexed by the universe level. We omit these indices.
- Specialized notation from HoTT-Agda will be replaced by traditional notation wherever the result may be assumed to be suitably unambiguous. This means, for example, that the $A \rightarrow^{G} B$ notation in HoTT-Agda used to denote a group homomorphism will be replaced by $A \rightarrow B$ whenever A and B are abelian groups, and the origins of the group operations $(+)$ will be implied by context. For when we need to emphasize that something is only known to be a map of sets, we will use the notation $A \rightarrow^{Set} B$.
- Record notation will be desugared to ”Book HoTT” by replacing record accessors (”foo.attribute”) by (dependent) functions (”attribute(foo)”).

- The chain complexes in HoTT-Agda are *augmented* chain complexes. For our purposes, it's much more convenient to deal with unaugmented chain complexes, so we provide our own definitions.

6.1.1 Graded Abelian Groups

Now, we begin by defining a type for $\mathbb{N}$-graded abelian groups:

$$\mathbb{N}GradedAbGroup :\equiv \mathbb{N} \to AbGroup$$

One particularly elegant thing about this definition is that equality of $GradedAbGroup$s reduces (by $funext$) to exhibiting a group isomorphism for any arbitrary grade. We define graded abelian group homomorphisms by:

$$GradedHom : GradedAbGroup \to GradedAbGroup \to \mathcal{U}$$

$$GradedHom(C, D) :\equiv \prod_{n:\mathbb{N}} C(n) \to D(n)$$

Defining the usual operations of composition, identity, and addition/subtraction on $GradedHom$s is routine.

6.1.2 Local Nilpotency and Alternating Geometric Sums

When we later define what it means for a perturbation to be locally nilpotent, the following definition will be useful:

Definition 6.1. Local Nilpotency An endomorphism ϕ of a graded abelian group X is *locally nilpotent* if there is a witness in the type family

$$localNilpotencyBound : \Big(\sum_{X:GradedAbGroup} GradedHom(X, X) \Big) \to \mathcal{U}$$

$$localNilpotencyBound((X, \phi)) :\equiv \prod_{m:\mathbb{N}} \prod_{x:X(m)} \sum_{n:\mathbb{N}} \phi(m)^n(x) = 0$$

where $\phi(m)^n$ denotes the nth iterate of the abelian group homomorphism $\phi(m)$.

It's important to note that for any (X, ϕ), $localNilpotencyBound(X, \phi))$ is *not* a mere proposition. Importantly, inhabitants of the type family carry information about an *explicit* bound on how big n needs to be to ensure that $\phi(m)^n(x) = 0$. By Exercise 3.19 of the HoTT book, to construct such a bound in principle, it would suffice to know an inhabitant of $||localNilpotencyBound(X, \phi))||$ so long as X has decidable equality, but this is impractical for two reasons:

- The witness to the type $||localNilpotencyBound(X, \phi))|| \to localNilpotencyBound(X, \phi)$ has a terrible computational cost – the bounds are derived by an exhaustive search on values of n, starting from zero! This is unsuitable for our intended application.

- Later, we may be interested in reductions where the top chain complex *does not* have a witness to decidable equality, but the bottom complex *does*.

So in our applications (like the computation of the total complex of a double complex) we will need to provide explicit bounds on the number of summands in terms such as ψ. This will turn out to not be too difficult.

Remark: We will make use of the fact that if we have a term of type $localNilpotencyBound((X, \phi))$ as above, then it's also straightforward to obtain a term of type:

$$\prod_{m:\mathbb{N}} \prod_{x:X(m)} \sum_{n:\mathbb{N}} \prod_{k:\mathbb{N}} k \geq n \to \phi(m)^k(x) = 0$$

since the identity is preserved under a group homomorphism.

Then, working towards expressing alternating geometric sums, we can write the following term using recursion on $\mathbb{N}$:

$$G : \prod_{X:GradedAbGroup} \prod_{\phi:GradedHom(X,X)} \prod_{m:\mathbb{N}} \big(\mathbb{N} \to (X(m) \to X(m))\big)$$

$$G(X, \phi, m, 0) :\equiv 1_{X(m)}$$

$$G(X, \phi, m, S(k)) :\equiv 1_{X(m)} - \phi(m) \circ G(X, \phi, m, k)$$

By an easy induction, we can prove that for all $m : \mathbb{N}$,

$$G(X, \phi, m, S(k)) = G(X, \phi, m, k) + (-1)^{S(k)}\phi(m)^{S(k)}$$

Hence, if $\phi(m)^{S(k)}(x) = 0$, $G(X, \phi, m, S(k))(x) = G(X, \phi, m, k)(x)$, or more generally (by a simple induction)

$$\prod_{j:\mathbb{N}} \Big((j \geq k) \to G(X, \phi, m, j)(x) = G(X, \phi, m, k)(x)\Big)$$

This also implies that

$$G(X, \phi, m, k)(x) = x - \phi(m)(G(X, \phi, m, k)(x))$$

Lemma. Alternating Geometric Sum of a Locally Nilpotent Endomorphism

Suppose that we have

$$X : GradedAbGroup, \phi : GradedHom(X, X), B : localNilpotencyBound(X, \phi)$$

Then, we can construct a graded map:

$$\phi^{\Sigma^\infty} : \prod_{m:\mathbb{N}} \big(X(m) \to^{Set} X(m)\big)$$

$$\phi^{\Sigma^\infty}(m)(x) :\equiv G(X, \phi, m, pr_1(B(m, x)))(x)$$

which is actually an abelian group endomorphism in every degree, and hence, this yields a graded abelian group endomorphism. Furthermore, we have:

$$\phi^{\Sigma^\infty} = 1_X - \phi \circ \phi^{\Sigma^\infty}$$

$$\phi^{\Sigma^\infty} = 1_X - \phi^{\Sigma^\infty} \circ \phi$$

Proof. The proof that the identity is preserved in every degree is simple, since at the end of the day, we will be applying some kind of group homomorphism to the identity. For composition, suppose that we have $x, y : X(m)$. From our local nilpotency bound B, we can derive:

$$\prod_{j:\mathbb{N}} \Big((j \geq pr_1(B(m,x))) \to G(X,\phi,m,j)(x) = G(X,\phi,m,pr_1(B(m,x)))(x) \Big)$$

$$\prod_{j:\mathbb{N}} \Big((j \geq pr_1(B(m,y))) \to G(X,\phi,m,j)(y) = G(X,\phi,m,pr_1(B(m,y)))(y) \Big)$$

$$\prod_{j:\mathbb{N}} \Big((j \geq pr_1(B(m,x+y))) \to G(X,\phi,m,j)(x+y) = G(X,\phi,m,pr_1(B(m,x+y)))(x+y) \Big)$$

Let

$$k_{max} :\equiv max\{pr_1(B(m,x)), pr_1(B(m,y)), pr_1(B(m,x+y))\}$$

Then,

$$G(X,\phi,m,k_{max})(y) = \phi^{\Sigma^\infty}(m)(y)$$

$$G(X,\phi,m,k_{max})(x) = \phi^{\Sigma^\infty}(m)(x)$$

$$G(X,\phi,m,k_{max})(x+y) = \phi^{\Sigma^\infty}(m)(x+y)$$

Finally, since $G(X,\phi,m,k_{max})$ is a group endomorphism,

$$\phi^{\Sigma^\infty}(m)(x) + \phi^{\Sigma^\infty}(m)(y) = \phi^{\Sigma^\infty}(m)(x+y)$$

as desired.

We may obtain the first recursive formula above by applying $funext$ so that we only need show it when applied to an arbitrary choice of $m : \mathbb{N}, x : X(m)$, whence it reduces to our recursive definition of G. For the second recursive formula, note that the first formula implies:

$$\phi^{\Sigma^\infty} \circ \phi = \phi - \phi \circ \phi^{\Sigma^\infty} \circ \phi$$

$$1 - \phi^{\Sigma^\infty} \circ \phi = 1 - \phi + \phi \circ \phi^{\Sigma^\infty} \circ \phi = 1 - (1_X - \phi \circ \phi^{\Sigma^\infty}) \circ \phi = 1 - \phi^{\Sigma^\infty} \circ \phi$$

and so indeed, both of our desired formulas hold.

□

We'll call the abelian group homomorphism obtained from this lemma $AltGeomSum(X, \phi, B)$.

6.1.3 Chain Complexes

Then, we define a type family of differentials on $\mathbb{N}$-graded abelian groups to obtain a straightforward definition of chain complexes:

$$BoundaryMap : GradedAbGroup \to \mathcal{U}$$

$$BoundaryMap(G) :\equiv \sum_{\partial:\prod_{n:\mathbb{N}} G_{S(n)} \to G_n} \prod_{k:\mathbb{N}} \partial_k \circ \partial_{S(k)} = 0$$

$$ChainComplex :\equiv \sum_{C:GradedAbGroup} BoundaryMap(C)$$

Then, for convenience, we will say that if $D : BoundaryMap(G)$, $\partial^D :\equiv pr_1(D)$. We also define the notation:

$$\_\_ : ChainComplex \to \mathbb{N} \to AbGroup$$

$$C_n :\equiv pr_1(C)(n)$$

$$\partial^{\_}_{\_} : \prod_{C:ChainComplex} \prod_{n:\mathbb{N}} C_{S(n)} \to C_n$$

$$\partial^C_n :\equiv pr_1(pr_2(C))(n)$$

Then, transcribing the usual definition of a chain map:

$$ChainMap : ChainComplex \to ChainComplex \to \mathcal{U}$$

$$ChainMap(C, D) :\equiv \sum_{\psi:GradedHom(C,D)} \prod_{k:\mathbb{N}} \psi(k) \circ \partial^C_k = \partial^D_k \circ \psi(S(k))$$

If $f : ChainMap(C, D)$, we will write f_k for $pr_1(f)(k)$. Showing that we can define $\circ$ and 1 on $ChainMap$s to yield a category of chain complexes is straightforward, so we don't waste space on it here.

Then, chain homotopies between chain maps may be defined as:

$$ChainHtpy : \prod_{C,D:ChainComplex} ChainMap(C, D) \to ChainMap(C, D) \to \mathcal{U}$$

$$ChainHtpy(C, D, f, g) :\equiv \sum_{h:\prod_{n:\mathbb{N}} C_n \to D_{S(n)}} \prod_{k:\mathbb{N}} \partial^D_{S(k)} \circ h_{S(k)} + h_k \circ \partial^C_k = f_{S(k)} - g_{S(k)}$$

Similarly to the case with chain maps, if we have an $h : ChainHtpy(C, D, f, g)$, we will write h_k for $pr_1(h)(k)$.

6.2 Reductions

With those general definitions out of the way, we can now focus in on reductions by starting with the general case of chain retractions:

Definition 6.2. Chain Retractions

$$ChainRetract : ChainComplex \to ChainComplex \to \mathcal{U}$$

$$ChainRetract(X, A) :\equiv \sum_{i:ChainMap(A,X)} \sum_{r:ChainMap(X,A)} r \circ i = 1_A$$

If $R : ChainRetract$, we'll use the notation i^R and r^R for $pr_1(R)$ and $pr_1(pr_2(R))$, respectively.

Note that if we have a chain retraction R from X to A, then we have an equality $Im(i^R \circ r^R) =_{ChainComplex} A$ by univalence, and from this, we can also extract an equality on the underlying graded abelian groups by functorial application of pr_1.

Then, specializing to deformation retractions, we have:

Definition 6.3. Type of Deformation Retractions

$$DefRetract : ChainComplex \to ChainComplex \to \mathcal{U}$$

$$DefRetract(X, A) :\equiv \sum_{R:ChainRetract(X,A)} ChainHtpy(X, X, 1_X, i^R \circ r^R)$$

If $R : DefRetract$, we will carry over the notation i^R and r^R from before by precomposition with pr_1 and will define h^R_k to be $pr_2(R)_k$. Finally,

Definition 6.4. Type of Reductions A *reduction* is any inhabitant of the type family:

$$Reduction : ChainComplex \to ChainComplex \to \mathcal{U}$$

$$Reduction(X, A) :\equiv \sum_{R:DefRetract(X,A)} \prod_{n:\mathbb{N}} (h^R_{S(n)} \circ h^R_n = 0) \times (h^R_n \circ i^R_n = 0) \times (r^R_{S(n)} \circ h^R_n = 0)$$

Once again, we will carry over the notation from earlier by precomposition with pr_1.

6.3 Perturbations and the Type of the BPL

Using the machinery above, since graded abelian group homomorphisms are closed under subtraction, instead of talking explicitly about perturbations of the differential on some chain complex C, we can talk about the resulting *perturbed differential* on C.

Definition 6.5. Type of Perturbed Differentials

$$PerturbedBoundary : ChainComplex \to \mathcal{U}$$

$$PerturbedBoundary(X) :\equiv BoundaryMap(pr_1(X))$$

Note that we can easily construct a map:

$$Perturb : \prod_{X:ChainComplex} PerturbedBoundary(X) \to ChainComplex$$

$$Perturb(X, P) :\equiv (pr_1(X), P)$$

Then, we may define

Definition 6.6. Local Nilpotency of Reduction Perturbation

$$LocallyNilpotent : \prod_{X,A:ChainComplex} Reduction(X, A) \to PerturbedBoundary(X) \to \mathcal{U}$$

$$LocallyNilpotent(X, A, R, P) :\equiv localNilpotencyBound((\Sigma(pr_1(X)), \overrightarrow{\Sigma}(h^R) \circ (\overleftarrow{\Sigma}(\partial^P) - \overleftarrow{\Sigma}(\partial^X))))$$

Where $\Sigma : AbGroup \to AbGroup$ denotes the suspension of a graded abelian group, and where

$$\overrightarrow{\Sigma} : (\prod_{n:\mathbb{N}} X_n \to X_{S(n)}) \to GradedHom(X, \Sigma(X))$$

$$\overleftarrow{\Sigma} : (\prod_{n:\mathbb{N}} X_{S(n)} \to X_n) \to GradedHom(\Sigma(X), X)$$

are natural identifications between degree boosting/degree deflating maps and graded homomorphisms to/from suspensions.

From this, we can write down the type of the BPL:

Definition 6.7. Type of the BPL

$$BPL : \prod_{X,A:ChainComplex} \prod_{R:Reduction(X,A)} \prod_{P^X:PerturbedBoundary(X)} \Bigg(LocallyNilpotent(X, A, R, P^X) \to \sum_{P^A:PerturbedBoundary(A)} Reduction(Perturb(X, P^X), Perturb(A, P^A))\Bigg)$$

6.4 Proof of the BPL

Proof. Let

$$X, A : ChainComplex, R : Reduction(X, A)$$

$$P^X : PerturbedBoundary(X), B : LocallyNilpotent(X, A, R, P^X)$$

Define the type family:

$$TargetedReduction : GradedAbGroup \to \mathcal{U}$$

$$TargetedReduction(Z) :\equiv \sum_{\partial^Z : BoundaryMap(Z)} Reduction(Perturb(X, P^X), Perturb(Z, \partial^Z))$$

Then, let

$$\begin{aligned}
&\delta = \overleftarrow{\Sigma}(\partial^{P^X} - \partial^X) && : GradedHom(\Sigma(X), X) \\
&h = \overrightarrow{\Sigma}(h^R) && : GradedHom(X, \Sigma(X)) \\
&\phi = AltGeomSum(\Sigma(X), h \circ \delta, B) && : GradedHom(\Sigma(X), \Sigma(X)) \\
&\psi = 1 - \delta \circ \phi \circ h \quad : && : GradedHom(X, X) \\
&\bar{h} = h \circ \psi \quad : && : GradedHom(X, \Sigma(X))
\end{aligned}$$

(here, X is written in place of $pr_1(X)$, since we are ignoring the chain complex structure)

Gathering all of our witnesses, we can see that

$$\psi = 1 - \delta \circ \phi \circ h = 1 - \delta \circ (1 - h \circ \delta \circ \phi) \circ h = 1 - \delta \circ h + \delta \circ h \circ \delta \circ \phi \circ h$$

$$= 1 - \delta \circ h \circ (1 - \delta \circ \phi \circ h) = 1 - \delta \circ h \circ \psi$$

Whence all of the conditions at the end of 5.1.2 follow, and so the conditions at the beginning of 5.1.3 also hold for the usual definitions of π and $\bar{\pi}$ (as graded abelian group homomorphisms), so we obtain the equality $P : Im(\bar{\pi}) =_{GradedAbGroup} Im(\pi)$ in the diagram from 6. The equality $Q : Im(\pi) =_{GradedAbGroup} pr_1(A)$ follows from the fact that A is a retract of X from our reduction R. Then, we have an element:

$$I :\equiv (\partial^{P^X}|_{Im(\bar{\pi})}, \rho) : TargetedReduction(Im(\bar{\pi}))$$

For ρ given by a term representing the reduction described at the beginning of 6.

Then $P \cdot Q : Im(\bar{\pi}) =_{GradedAbGroup} pr_1(A)$, so:

$$F :\equiv transport^{TargetedReduction}(P \cdot Q, I) : TargetedReduction(pr_1(A))$$

We can expand the definition of $TargetedReduction(pr_1(A))$ to see:

$$F : \sum_{\partial^A : BoundaryMap(pr_1(A))} Reduction(Perturb(X, P^X), (pr_1(A), \partial^A))$$

but then, setting $P^A :\equiv \partial^A$ this type is definitionally equal to:

$$F : \sum_{P^A : PerturbedBoundary(A)} Reduction(Perturb(X, P^X), Perturb(A, P^A))$$

so F is a term of exactly the type we needed to return from the BPL. $\square$

7 Properties of the Proof in HoTT

The proof above made use of $funext$ and univalence in two crucial places for the sanity of the proof: The first was in our use of $funext$ to simplify our recursive formulas for alternating geometric sums of locally nilpotent homomorphisms so that we could avoid referring to particular elements throughout the remainder of the proof. The second was in our use of the univalence axiom to construct *equalities* between graded abelian groups so that we could use $transport$ instead of proving a boring isomorphism lemma about reductions. The absence of both of these improvements is probably the reason why the concept for this proof did not find its way into earlier formalizations of the BPL based on fully constructive foundations. Interestingly, the proof in Isabelle/HOL (seen in "A Mechanized Proof of the Basic Perturbation Lemma by Jess Aransay Clemens Ballarin Julio Rubio") adopts this strategy, despite the pain induced by the explicit isomorphisms.

8 Appendix: The Suspension as a Homotopy Pushout

One of the definitions briefly mentioned earlier for the suspension ΣX is as a pair of two poles together with a natural assignment of paths from both poles to every point in X. Using this definition, and recognizing that the terminal object $\star$ in the category Top is any point by itself, we can say that ΣX is the homotopy pushout of the span $\begin{tikzcd} X \arrow[r] \arrow[d] & \star \\ \star & \end{tikzcd}$ That is, ΣX is such that:

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \star \arrow[d, "N"] \\
\star \arrow[r, "S"] & \Sigma X
\end{tikzcd}$$

commutes up to a (really, arbitrarily-chosen) homotopy h between the constant map at the north pole and the constant map at the south pole, and for any $Z : Top$ such that

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \star \arrow[d, "\bar{N}"] \\
\star \arrow[r, "\bar{S}"] & Z
\end{tikzcd}$$

commutes up to a homotopy $\bar{h}$ between $\bar{N}$ and $\bar{S}$, we have a map $\phi : \Sigma X \to Z$ such that the whole diagram

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & \star \arrow[d, "N"'] \arrow[ddr, bend left, "\bar{N}"] & \\
\star \arrow[r, "S"] \arrow[drr, bend right, "\bar{S}"'] & \Sigma X \arrow[dr, "\phi"] & \\
 & & Z
\end{tikzcd}$$

commutes up to homotopy. This definition also works in any category where we have a terminal object and a good notion of homotopy.

The proof that the classical suspension satisfies this property is easy, since any homotopy between $\bar{N} \circ \star_X$ and $\bar{S} \circ \star_X$ is a map $h : I \times X \to Z$ such that $h_0(x) = \bar{N}(\star)$ and $h_1(x) = \bar{S}(\star)$, so we can factorize through the quotient to get the desired map ϕ with domain ΣX.

In $Ch(A)$, the terminal object is instead $\mathbf{0}$, but for the suspension of a chain complex X, we'll stick with the ΣX constructed in section 1.2. From here, we need a homotopy between $\mathbf{0} : X \to \Sigma X$ and $\mathbf{0}$, as the first diagram turns into:

$$\begin{tikzcd} X \arrow[r, "\mathbf{0}"] \arrow[d, "\mathbf{0}"] & \mathbf{0} \arrow[d, "\mathbf{0}"] \\ \mathbf{0} \arrow[r, "\mathbf{0}"] & \Sigma X \end{tikzcd}$$

So we need a chain homotopy $\{h_n : X_n \to [\Sigma X]_{n+1}\}_{n \in \mathbb{N}}$. Pick the shift operator s to fulfill this role. Then, to show this gives a chain homotopy, we need to verify that:

$$\partial^{\Sigma X} \circ s + s \circ \partial^X = 0$$

but we know that the shift operator behaves as the identity after identifying $[\Sigma X]_{n+1}$ with X_n, and so by the definition of the boundary operator on the suspension, we see that this reduces to $-\partial^X + \partial^X = 0$. This explains that mysterious sign change in the differential.

Now, we only need to verify that the universal property of the homotopy pushout holds. Suppose we have a $Z : Ch(A)$, and another chain homotopy $\{\bar{h}_n : X_n \to Z_{n+1}\}_{n \in \mathbb{N}}$ such that $\partial^Z \circ \bar{h} + \bar{h} \circ \partial^X = 0$. All we need to do (due to the infectuous $\mathbf{0}$ in the rightmost and bottommost triangles of the pushout diagram) is to show that there's a chain map ϕ from ΣX to Z. Concretely, this means that we need:

$$\phi \circ \partial^{\Sigma X} = \partial^Z \circ \phi$$

so, expanding the definition of ΣX,

$$\partial^Z \circ \phi + \phi \circ \partial^X = 0$$

but this condition is satisfied if we let $\phi = \bar{h}$ from before. So it's reasonable to call ΣX the suspension of X.